Prop (Clairaut's Theorem): 10/4/21

If $f(x,y)$ has continuous second-order partial derivatives on an open Disk, D, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ on D.

Notation: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$

$$f_{xx} = f(x)_x = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[f]\right] = \frac{\partial^2 f}{(\partial x)^2}$$

$$f_{xy} = f(x)_y = \frac{\partial}{\partial y}[f_x] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}[f]\right] = \frac{\partial^2 f}{\partial y \partial x}$$

$\uparrow$

notation is nicer for today....

Pf: Let $f(x,y)$ have continuous second-order mixed partial derivatives on some open disk D and suppose $(a,b) \in D$.

Let $\Delta(h) = \left(f(a+h, b+h) - f(a+h, b)\right) - \left(f(a, b+h) - f(a,b)\right)$

for all $h \neq 0$ where $(a+h, b+h), (a+h, b), (a, b+h)) \in D$.

Let $\alpha(x) := f(x, b+h) - f(x, b)$ and notice

$\Delta(h) = \alpha(a+h) - \alpha(a)$ for h fixed,

we can apply the MVT to obtain c_h satisfying $|a - c_h| \le |h|$
and $\alpha'(c_h)h = \alpha(a+h) - \alpha(a)$ thus
$\Delta(h) = \alpha(a+h) - (\alpha(a)) = h\alpha'(c_h) = h(f_x(c_h, b+h) - f_x(c_h, b))$

Letting $\rho(y) = f_x(c_h, y)$, we see again by MVT there is d_h satisfying $|b - d_h| \le |h|$ and $\rho'(d_h)h = f_x(c_h, b+h) - f_x(c_h, b)$

Thus $\Delta(h) = h(f_x(c_h, b+h) - f_x(c_h, b)) = h(h\rho'(d_h))$
$= h^2 f_{xy}(c_h, d_h)$

If we rearrange $\Delta(h) = (f(a+h, b+h) - f(a, b+h))$
$- (f(a+h, b) - f(a, b))$

we can repeat the same argument (working with y first) to obtain γ_h, ρ_h satisfying $|a - \gamma_h| \le |h|$, $|b - \rho_h| \le |h|$ and $\Delta(h) = h^2 f_{yx}(\gamma_h, \rho_h)$ for all fixed h.
$**$

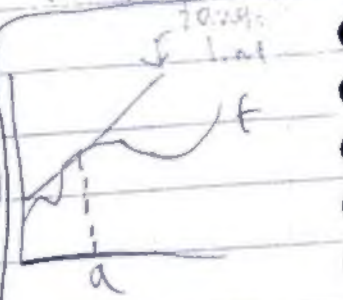
Notice $\lim_{h\to 0} (c_h, d_h) = (a, b) = \lim_{h\to 0} (\gamma_h, \rho_h)$
by construction, thus we compute:

$$f_{xy}(a,b) = f_{xy}\left(\lim_{h\to 0}(c_h, d_h)\right)$$
continuity
$$= \lim_{h\to 0} f_{xy}(c_h, d_h)$$
by $*$
$$= \lim_{h\to 0} \frac{\Delta(h)}{h^2}$$
$$= \lim_{h\to 0} f_{yx}(\gamma_h, \rho_h)$$
$$= f_{yx}\left(\lim_{h\to 0}(\gamma_h, \rho_h)\right)$$
$$= f_{yx}(a,b)$$

☺

§14.4: Linear Approximation of Multivariable Functions

Idea: In Calc I, we say the tangent line to f at a, "well-approximates" f near (a, f(a))



In Calc 3, we use a tangent plane (hyperplane) instead (again minimizing the tangent approximation...)

sufficient for 2 variable

only necessary if f has more than 2 variables

as $x \to a$, the error approximation f with the tangent line gets to 0.

Small changes in input have change for output of f measured by the first derivatives.

In Calc I $f(x) \approx y = f(a) + f'(a)(x-a)$ near input a

In Calc III, these changes are measured by:

$$f(x,y) \approx z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Ex: Find tangent plane equation to $f(x,y) = x^2 + xy - y^2$ at $(4,1)$

Sol: Tangent plane has equation $z = f(a,b) + f_x(a,b)(x-a) - f_y(a,b)(y-b)$

So we compute $f(4,1) = 4^2 + (4)(1) - 1^2 = 19$

$f_x(x,y) = 2x+y$ $\qquad$ $f_y(x,y) = x-2y$

$f_x(4,1) = (2)(4)+1 = 9$ $\qquad$ $f_y(4,1) = 4-(2)(1) = 2$

Plane $= z = 19 + 9(x-4) + 2(y-1)$

EX: Compute the tangent plane to $f(x,y)=\frac{e^{x-y}}{x}$ at $(2,2,\frac{1}{2})$

Sol: The tangent plane has equation $z = f(a,b) + f_x(a,b)(x-a)$
$+ f_y(a,b)(y-b)$

We compute $f(2,2)=\frac{1}{2}$

$$f_x(x,y) = \frac{x e^{x-y} - e^{x-y}\cdot 1}{x^2} = e^{(x-y)}\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$$f_y(x,y) = -\frac{e^{x-y}}{x}$$

$$f_x(2,2) = e^0\left(\frac{1}{2} - \frac{1}{2^2}\right) = \frac{1}{4}$$

$$f_y(2,2) = -\frac{e^0}{2} = -\frac{1}{2}$$

Planar Tangent $= z = \frac{1}{2} + \frac{1}{4}(x-2) - \frac{1}{2}(y-2)$

In Calc I, we also thought from the perspective of differentials:

$$\Delta f \approx (a)\,\Delta x \quad \text{at } a = x$$

Changing f from Changing x

Change in x

For functions of 2 variables:

$$\Delta f \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

for small perturbation from (a,b)

In Calc I, we replace Δ's by symbols and associated eqn

$$df = f'(x)dx \quad \text{i.e.} \quad df = \frac{df}{dx}dx$$

<u>Def^n</u>: The <u>total differential</u> of function f of variables $x_1, \ldots, x_n$ is

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n$$

represents Δf but not exactly Δf.

symbols

<u>Ex</u>: Compute the total differential of $f(x,y,z) = \frac{\log(x-3y)}{z}$

<u>Sol</u>: We compute:

$$f_x(x,y,z) = \frac{1}{z}\cdot\frac{1\cdot 1}{x-3y} = \frac{1}{(x-3y)z}$$

$$f_y(x,y,z) = \frac{1}{z}\cdot\frac{1\cdot(-3)}{x-3y} = \frac{-3}{(x-3y)z}$$

$$f_z(x,y,z) = \frac{\ln(x-3y)}{z^2}$$

$$df = f_x dx + f_y dy + f_z dz = \frac{1}{(x-3y)z}dx - \frac{3}{(x-3y)z}dy - \frac{\ln(x-3y)}{z^2}dz$$

<u>Ex</u>: estimate Δf from $(4,1,1)$ to $(4.5, 1.5, 0.5)$ (previous example)

<u>Sol</u>: $\Delta f \approx df$

$\Delta x_i \approx dx_i$

$$\Delta f \approx f_x(4,1,1)\Delta x + f_y(4,1,1)\Delta y + f_z(4,1,1)\Delta z$$

$$\Delta f = \frac{1}{(1)(1)}(4.5-4) - \frac{3}{(1)(1)}(1.5-1) - \frac{\ln(1)}{(1)^2}(2.5-1)$$

$$\boxed{\Delta f = -1}$$